

Courant Institute of
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Construction of Steady-State
Hydromagnetic Dynamos

II. The Spherical Conductor

Stephen Childress

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Construction of Steady-State Hydromagnetic
Dynamoes. II. The Spherical Conductor

STEPHEN CHILDRESS

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Abstract

The steady-state kinematic dynamo problem in a homogeneous sphere with concentric spherical fluid core is considered. The existence of a class of solenoidal dynamos satisfying a no-slip condition on the core boundary is proved by perturbing the discrete spectrum of a simpler self-adjoint comparison problem. The dynamos are of the form $\vec{q} = \vec{q}^{(1)} + \vec{q}^{(2)} + \vec{q}^{(3)}$ where $\vec{q}^{(1)}$ is spatially periodic, $\vec{q}^{(2)}$ is zero except in a neighborhood of the boundary of the core, and $\vec{q}^{(3)}$ is an arbitrary sufficiently small motion.

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1. Introduction.

In this report we study the steady-state induction of an electromagnetic field by a fluid motion in a spherical conductor of radius unity having a concentric spherical core of radius a . The principal result of the investigation is an explicit construction of a class of bounded hydromagnetic dynamos of the form

$$\begin{aligned}\vec{q} &= -\varepsilon \nabla \times \omega \vec{\nabla} \left(\frac{\vec{x}}{\varepsilon} \right) + \varepsilon \beta \vec{w}(\vec{x}) & 0 \leq x \leq a \\ &= \omega \vec{v} + \varepsilon \nabla \omega \times \vec{v} + \varepsilon \beta \vec{w}\end{aligned}\quad (1)$$

Here $\vec{x} = \{x_\alpha\}$, $x = |\vec{x}|$, β is any positive number, $\vec{w}$ is an arbitrary continuously differentiable solenoidal field, independent of ε and satisfying $\vec{w} = 0$ on $x = a$, $\vec{v}$ has Cartesian components $(\sin x_2 + \cos x_3, \sin x_3 + \cos x_1, \sin x_1 + \cos x_2)$, and the scalar function ω is smooth "cut-off" function, given e.g. by

$$\omega = \begin{cases} \varepsilon^{-\frac{1}{2}}, & x < a - \varepsilon \\ \frac{\varepsilon^{-\frac{1}{2}}}{\sqrt{\pi}} \int_{-\infty}^{p(x)} e^{-\xi^2} d\xi, & p(x) = \frac{a + \varepsilon/2 - x}{(x - a + \varepsilon)(a - x)}, \quad a - \varepsilon \leq x \leq a \end{cases} \quad (2)$$

Specifically, we shall show that motions of the form (1) are dynamos provided ϵ is sufficiently small. The field $\epsilon^{-\frac{1}{2}}\vec{v}(\vec{x}/\epsilon)$ is also a periodic dynamo in the sense of Part I,¹ satisfying a Beltrami condition $\nabla \times \vec{v}(\vec{x}) = -\vec{v}(\vec{x})$. For small ϵ , the term $\omega\vec{v}$ in (1) essentially dominates the generative process in the conductor, so that the present analysis is closely related to the construction of periodic dynamos as discussed in I.

In particular, we find that as $\epsilon \rightarrow 0$ the large scale electromagnetic fields induced by (1) tend uniformly to solutions of a certain self-adjoint "comparison" eigenvalue problem, comparison core equations being analogous to the partial-differential form of the limiting compatibility problem in the periodic case. For a sphere the comparison problem is easily solved and the spectrum of comparison eigenvalues (which is discrete) contains a symmetric set of real point eigenvalues of odd multiplicity accumulating at the origin. Formulated operationally, a modified second iterate of the exact induction

1. S. Childress, "Construction of Steady-State Hydromagnetic Dynamos.I. Spatially-Periodic Fields." (Unpublished), hereafter referred to as I.

operator is close (in an appropriate norm) to the self-adjoint comparison operator. It is then possible to develop a perturbation of the comparison operator and obtain real eigenvalues of the exact operator which are close to one of the square roots of each positive comparison eigenvalue. A second class of dynamos and electromagnetic eigensolutions, associated with negative comparison eigenvalues, can be obtained by reflection, since in a sphere $-\vec{q}(-\vec{x})$ is a dynamo if $\vec{q}(\vec{x})$ is a dynamo. For simplicity we shall restrict the present discussion to the motions (1).

Mathematically, the significant feature of the analysis is the use of a non-inner product ("maximum") norm to achieve smallness of the operator perturbation. The perturbational analysis is then carried out in Banach space. We emphasize this point since the topology most commonly utilized in dynamo theory is the topology of the Hilbert space norm.^{2,3} For the motions (1), the natural Hilbert spaces appear to be too "wide". A magnetic field which is bounded in the inner-product norm may interact with a motion of the form (1) to induce a magnetic field which becomes arbitrarily large in the same norm as $\epsilon \rightarrow 0$.

2. W.M. Elsasser, Phys. Rev. 69, 106 (1946).

3. G. Backus, Ann. Phys. 4, 372 (1958).

This need not be the case, however, on suitably small Banach spaces when the norm includes the maximum value of the field and its first derivatives over the closure of the core. Boundedness of the exact operator is here viewed as a desirable, rather than necessary property for a perturbational theory. (Indeed, the important issue is the smallness of the perturbational operator, and in the present problem these two norms are distinct, cf. Sections 4,5.) Without a uniformly bounded exact operator the choice of comparison operator (which then depends upon ε) is likely to be considerably more difficult.

The above reasoning can be illustrated by a simple one dimensional model of our basic induction operator. Let $f(x)$ satisfy $N_\gamma(f) < \infty$ where the Hilbert space norm is

$$N_\gamma(f) = \left(\int_0^1 f^2 dx \right)^{\frac{1}{2}} + \varepsilon^\gamma \left(\int_0^1 f_x^2 dx \right)^{\frac{1}{2}}$$

and consider the operator ℓ defined by

$$\ell f = \varepsilon^{-\frac{1}{2}} \int_0^1 k(x,y) \sin \frac{y}{\varepsilon} f(y) dy$$

with $k(x,y)$ independent of ε and positive and continuously differentiable on $[0,1]$. Consider the

function

$$g(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \sqrt{\varepsilon}, \quad \frac{1}{2} + \sqrt{\varepsilon} \leq x \leq 1 \\ \varepsilon^{-1/4} \sin \frac{x}{\varepsilon} & \frac{1}{2} - \sqrt{\varepsilon} < x < \frac{1}{2} + \sqrt{\varepsilon} \end{cases}$$

Clearly if $\gamma \geq 1$, $N_\gamma(g) = O(1)$ as $\varepsilon \rightarrow 0$. On the other hand

$$N_\gamma(\ell f) \geq C \varepsilon^{-1/4}, \quad \gamma > 0$$

for some $C > 0$ and ε smaller than some positive ε_0 . However, if we adopt the maximum norm

$$||f||_{\frac{1}{2}} = \max_{[0,1]} |f| + \varepsilon^{\frac{1}{2}} \max_{[0,1]} \left| \frac{df}{dx} \right|$$

it is not difficult to see that on the corresponding Banach space

$$||\ell|| = O(1) \text{ as } \varepsilon \rightarrow 0.$$

The physical basis of the regenerative cycle utilizing periodic fields of degree two has been discussed in I. A related idea is implicit in the time-dependent cycle proposed by Parker.⁴ The notion

4. E. N. Parker, *Astrophys. J.* 122, 293 (1955)

of a "scale separation" is in fact rather common in dynamo theory. In Backus' time-dependent dynamo it occurs as a separation of the cycle into well defined periods of motion and (sufficiently long) periods of "stasis". In Herzenberg's two-sphere steady state dynamo, the small parameter is the ratio of sphere radius to conductor radius.⁵ The present results may, therefore, be viewed as an extension of such kinematic models to essentially "fluid" motions which cannot be built up from a finite collection of rigid rotators.

2. Formulation

We consider a homogeneous spherical conductor of radius unity having a concentric spherical core of radius $a < 1$. For simplicity we assume that the solid and fluid parts of the conductor have the same (constant) electrical conductivity, and that the core fluid is incompressible. We assume further that the fluid moves with velocity $\lambda^{-1}\vec{q}(\vec{x};\varepsilon)$ where $\vec{q}$ is given by (1), (2), λ being a non-zero eigenparameter. Our object is to prove that for ε sufficiently small there is at least one real eigenvalue λ such that the magnetohydrodynamic induction equations have a non-trivial

5. A. Herzenberg, Phil. Trans. Roy. Soc. London 250, 543 (1958)

solution defined in all space and satisfying the required boundary and regularity conditions. In dimensionless form the equations are ($\vec{h}$ = magnetic field, $\vec{e}$ = electric field, ϕ = electric potential)

$$\nabla \times \vec{h} = \vec{e} + \lambda^{-1} \vec{q} \times \vec{h} \quad 0 \leq x < a \quad (3a)$$

$$\nabla \times \vec{h} = \vec{e} \quad a < x < 1 \quad (3b)$$

$$\nabla \times \vec{h} = 0 \quad x > 1 \quad (3c)$$

$$\nabla \times \vec{e} = 0 \quad (\vec{e} = \nabla \phi) \quad (3d)$$

$$\nabla \cdot \vec{h} = 0 \quad (3e)$$

and the boundary and regularity conditions are

$$\vec{e} \text{ continuous, } \vec{h} \text{ continuously differentiable} \quad (4a)$$

if $x \neq a, 1$

$$\vec{h} \text{ and } \vec{x} \times \vec{e} \text{ continuous on } x = a, 1 \quad (4b)$$

$$\lim \vec{h} = 0, \lim \vec{e} = 0 \quad \text{as } x \rightarrow \infty \quad (4c)$$

We begin our analysis of this problem by reformulating (3), (4) as an integral equation. Let S denote the region $x < a$, B the core boundary $x = a$. Let $\mathbf{K}(\vec{x}, \vec{\xi})$, $\mathbf{E}(\vec{x}, \vec{\xi}) = \nabla_{\vec{x}} \vec{\phi}$ be respectively the magnetic and electric fields caused by a current distribution $\delta(\vec{x} - \vec{\xi})$ where $\vec{\xi} \in S$. By the superposition principle we then have as the formal integral equation replacing (3), (4)

$$\vec{Lh} \equiv \int_S \mathbf{K}(\vec{x}, \vec{\xi}) \cdot [\vec{q}(\vec{\xi}, \varepsilon) \times \vec{h}(\vec{\xi})] d\vec{\xi} = \lambda \vec{h}(\vec{x}), \quad \vec{x} \in S \quad (5)$$

The corresponding electric field potential is given by

$$\phi(\vec{x}) = \lambda^{-1} \int_S \vec{\phi}(\vec{x}, \vec{\xi}) \cdot [\vec{q}(\vec{\xi}, \varepsilon) \times \vec{h}(\vec{\xi})] d\vec{\xi} \equiv \lambda^{-1} M\vec{h} \quad (6)$$

For the sphere an explicit construction of $\mathbf{K}$, $\vec{\phi}$ can be given. Consider first the electric field. In S we have, in component form,

$$\nabla_x^2 \phi_\alpha = - \frac{\partial \delta}{\partial x_\alpha} \quad (7a)$$

Now (3d) and (4b) imply that the normal component of the current $\nabla \times \vec{h}$ vanishes on B , so that in S (7a) must be solved with the condition

$$\frac{\partial \phi_\alpha}{\partial x} \rightarrow 0 \text{ as } x \rightarrow 1. \quad (7b)$$

Thus, for $x < 1$

$$\phi_\alpha = - \frac{\partial N}{\partial \xi_\alpha} \quad (8a)$$

where N is the Neumann function for the unit sphere:

$$N(\vec{x}, \vec{\xi}) = \frac{1}{4\pi} \left[\frac{1}{r} + \frac{1}{\xi R} + \log \frac{2}{1-x\xi \cos \theta + \xi R} \right] \quad (8b)$$

where (see Figure 1)

$$r = |\vec{x} - \vec{\xi}| \quad , \quad \vec{x} \cdot \vec{\xi} = x\xi \cos \theta$$

$$\xi R = \left[x^2 \xi^2 - 2x \cdot \xi + 1 \right]^{\frac{1}{2}}$$

We may then extend $\vec{\phi}$ continuously into $x > 1$ by solving an exterior Dirichlet problem with boundary values on B given by (8).

To find the magnetic field, let $\vec{u}$ be an arbitrary unit vector and set

$$\vec{h}_u = \vec{h} \cdot \vec{u} = \frac{1}{4\pi} \nabla_x \left(\frac{1}{r} \right) \times \vec{u} + \vec{h}'_u$$

so that

$$\nabla_x \times \vec{h}'_u = \left[\vec{u} + \frac{1}{4\pi} \nabla_x \nabla_{\xi} \left(\frac{1}{r} \right) \right] \cdot \vec{u} = \mathbf{E}' \cdot \vec{u}$$

say. In a sphere the last equation may be inverted to give

$$\vec{h}'_u = - \int_0^1 t \left[\vec{x} \times \mathbf{E}(t\vec{x}, \vec{\xi}) \cdot \vec{u} \right] dt + \nabla_x \phi_1 \quad , \quad \nabla_x^2 \phi_1 = 0 \quad (9)$$

$$0 \leq x < 1$$

$$\vec{h}_u = \nabla_x \phi_2 \quad , \quad \nabla_x^2 \phi_2 = 0 \quad x > 1 \quad (10)$$

It is then easily shown that ϕ_2 and ϕ_1 can be represented by a double-layer potential on B, the distribution being determined by the first term on the right of (9). This construction uniquely determines **K** and **E**, and it follows from known properties of the potential that the non-singular parts of these tensors are regular over any closed sub-domain of $x < 1$, e.g. over the closure of the core.

In fact, from (9) the existence of **K**, **E** can be proved for any smooth, convex conductor. For the unit sphere, however, **K** has some particular properties of interest. In this case it can be shown that

$$\vec{h}_u' = \nabla_x \times (\vec{x}\psi) \quad x < 1 \quad (11)$$

where

$$\psi = -\vec{u} \cdot \nabla_{\xi} [\log (1 - \vec{x} \cdot \vec{\xi} + \xi R)] \quad (12)$$

Also

$$\frac{1}{4\pi} \nabla_x \left(\frac{1}{r} \right) \times \vec{u} = \nabla f_u + \nabla_x \times (\vec{x}g_u) \quad (13a)$$

where

$$f_u = \vec{u} \cdot \nabla_{\xi} \frac{1}{4\pi x} \log (x - \xi \cos \theta + r) \quad (13b)$$

$$g_u = - \frac{\vec{u} \cdot (\vec{x} \times \vec{\xi})}{xr} \left(\frac{1}{x - \xi \cos \theta + r} \right) \quad (13c)$$

It follows from (11) - (13) that $\nabla_x \times [x(g + \psi)] = 0$ when $x = 1$ and hence that

$$\vec{h}_u = \nabla f_u + \nabla_x [\vec{x}(g_u + \psi)] \quad x < 1 \quad (14a)$$

$$= \nabla f_u \quad x > 1 \quad (14b)$$

In arriving at (14) we have made essential use of the fact that for a sphere the outward normal of the boundary equals $\vec{x}/x$.

We next introduce the integral transformations

$$H\vec{h} = \int_S \mathbf{K} \cdot \vec{h} d\vec{\xi} \quad , \quad J\vec{h} = \int_S \vec{\phi} \cdot \vec{h} d\vec{\xi} \quad (15a)$$

and the comparison eigenvalue problem

$$H\vec{h} = \mu \vec{h} \quad \mu \neq 0 \quad (15b)$$

The significance of (15) in relation to (1) and (5) will be apparent later. We note here that H is symmetric (i.e. self-adjoint) and H^2 is compact on the real Hilbert space ~~of~~ of solenoidal vector field

$\vec{h}$ which are square integrable over the core, the associated inner product and norm being

$$(\vec{f}, \vec{g}) = \int_{x < a} \vec{f} \cdot \vec{g} \, d\vec{x}$$

$$||\vec{h}||_{\mathcal{H}} = (\vec{h}, \vec{h})^{\frac{1}{2}}$$

respectively. The compactness of H^2 follows from the fact that the iterated transformation can be expressed as a single integral transformation having a square-integrable kernel.⁶ To prove the self-adjointness of H we must make strong use of the representation (14) of K for the unit sphere. We shall show that

$$(\vec{f}, H\vec{g}) = (H\vec{f}, \vec{g}) \quad (16a)$$

for all $\vec{f}, \vec{g} \in \mathcal{H}$. Now by the definition of H , $\vec{h} = H\vec{f}$ is defined in all space and satisfies

$$\nabla \times \vec{h} = \begin{cases} \nabla J\vec{f} + \vec{f} & x < a \\ \nabla J\vec{f} & a < x < 1 \\ 0 & 1 < x \end{cases}$$

6. A. E. Taylor, Introduction to Functional Analysis (John Wiley and Sons, Inc. 1958), Chapter 5.

Therefore

$$\int_{\text{all space}} \nabla \times \vec{h} \cdot \vec{H} g d\vec{x} = \int_{x < 1} \nabla J \vec{F} \cdot \vec{H} g d\vec{x} + (\vec{F}, \vec{H} g)$$

Using now the vector identity $\vec{B} \cdot \nabla \times \vec{A} = \vec{A} \cdot \nabla \times \vec{B} + \nabla \cdot (\vec{A} \times \vec{B})$, the continuity of $H\vec{F}$, $H\vec{g}$ and obvious estimates for large x following (15), integration by parts gives

$$\int_{\text{all space}} \nabla \times \vec{h} \cdot \vec{H} g d\vec{x} = \int_{x < 1} \vec{h} \cdot \nabla J g d\vec{x} + (H\vec{F}, \vec{g})$$

Combining the last two equations we obtain

$$(\vec{F}, H\vec{g}) - (H\vec{F}, \vec{g}) = \int_{x < 1} [\nabla J \vec{g} \cdot H\vec{F} - \nabla J \vec{F} \cdot H\vec{g}] d\vec{x} \quad (16b)$$

On the other hand, from (14) we observe that there are scalar fields F, G , with $\nabla^2 F = 0$, such that

$$\int_{x < 1} \nabla J \vec{g} \cdot H\vec{F} d\vec{x} = \int_{x < 1} \nabla J \vec{g} \cdot \nabla F d\vec{x} + \int_{x < 1} \nabla J \vec{g} \cdot \nabla \times G \vec{x} d\vec{x}$$

But, since $\nabla^2 J \vec{g} = -\nabla \cdot \vec{g} = 0$ for $x < a$ and $\vec{x} \cdot \nabla J \vec{g} = -\vec{x} \cdot (\nabla \times H\vec{g}) = 0$ on $x = 1$ we have

$$\int_{x < 1} \nabla J \vec{g} \cdot \nabla F d\vec{x} = \int_{x=1} \vec{F} \cdot \vec{x} \cdot \nabla J \vec{g} ds = 0$$

Also

$$\int \nabla J \vec{g} \cdot \nabla \times G \vec{x} d\vec{x} = \int_{x=1} J \vec{g} (\nabla \times G \vec{x}) \cdot \vec{x} ds = 0$$

Thus the right-hand side of (16b) vanishes, which proves (16a).

In this report we are not going to develop the perturbational theory in the Hilbert space topology, for reasons discussed in Section 1. We shall definitely use the self-adjoint property of H , and at one point the compactness of H^2 (cf. Section 3), but the principal linear space we want to consider is not an inner product space.

We introduce the space $C^1 = C^1(\overline{S})$ of vector fields whose first derivatives are continuous in the closure of the core, $\overline{S}$. Because of the occurrence of the parameter ε in (1), (2), we will be interested in allowing a parameter dependence on ε in both the vector fields in C^1 and the operators on C^1 .

To make C^1 into a Banach space we introduce the family of norms (γ -norms)

$$||W||_{\gamma} = ||W|| + \varepsilon^{\gamma} ||\nabla W|| \quad 0 < \gamma \leq 1$$

where for any tensor $W = \{W_{\alpha\beta} \cdots \gamma\}$

$$||W|| = \max_{\alpha, \beta, \dots, \gamma} \left[\max_{\vec{S}} |W\alpha\beta\dots\gamma| \right]$$

We shall utilize the standard O and o symbols to express boundedness or smallness as $\varepsilon \rightarrow 0$. If $f(\varepsilon)$ is continuous in a neighborhood of $\varepsilon = 0$, $||\vec{h}||_{\gamma} \leq C |f(\varepsilon)|$ for some positive constant uniformly in some interval (ε_0 -interval) of the form $0 < \varepsilon \leq \varepsilon_0$, $\varepsilon_0 > 0$. Similarly $||\vec{h}||_{\gamma} = o(f)$ will mean that for any $\delta > 0$ there is an ε_0 -interval where $||\vec{h}||_{\gamma} < \delta |f(\varepsilon)|$ uniformly.

Given an operator $T(\varepsilon)$ on C^1 , we say that T is bounded (for fixed ε) if there is a number m such that $||T\vec{h}||_{\gamma} \leq m ||\vec{h}||_{\gamma}$ for all $h \in C^1$. The least upper bound on the set of all such m is called the γ -norm of T and is denoted by $||T||_{\gamma}$. We shall say that T is uniformly bounded if $||T||_{\gamma} = O(1)$ as $\varepsilon \rightarrow 0$, and that T is uniformly small if $||T||_{\gamma} = o(1)$ as $\varepsilon \rightarrow 0$.

We prove next that L defined by (5) is bounded on C^1 , and that H defined by (15a) is uniformly compact on C^1 . (Therefore H is bounded, and hence uniformly bounded on C^1 .) In Section 4 we shall show in addition that L is uniformly bounded on C^1 when $\gamma = \frac{1}{2}$ defines the norm. To show that L is bounded pointwise in ε in any γ norm we observe that

$$\int_S \mathbf{K} \cdot (\vec{q} \times \vec{h}) d\vec{\xi} = \frac{1}{4\pi} \int_S \frac{\nabla \times (\vec{q} \times \vec{h})}{r} d\vec{\xi} + \int_S \mathbf{F} \cdot \vec{q} \times \vec{h} d\vec{\xi}$$

after integration by parts, using $\vec{q} = 0$ on B . Here $\mathbf{F}$ denotes the regular part of $\mathbf{K}$. Since $\vec{q} \in C^1$ and $h \in C^1$ by hypothesis, the first term on the right can be differentiated and the result is continuous in $\bar{S}$. Since $\mathbf{F}$ is regular in $\bar{x}$ the conclusion follows. A similar argument can be applied, incidentally, to M .

To prove the uniform compactness of H it is sufficient to show that H maps a uniformly bounded sequence $\{\vec{h}_i, i = 1, 2, \dots, ||\vec{h}_i||_\gamma < M\}$ in C^1 into an equicontinuous family in C^1 . To see that this can be done write, as before,

$$H\vec{h} = \frac{1}{4\pi} \int_S \frac{\nabla \times \vec{h}}{r} d\vec{\xi} + \frac{1}{4\pi} \int_B \frac{\vec{h} \times \vec{h}}{r} ds + \int_S \mathbf{F} \cdot \vec{h} d\vec{\xi}$$

If we differentiate $H\vec{h}$ with $h \in C^1$, we see that the result can be expressed as the sum of the derivative of a volume potential of a bounded and integrable distribution, a single-layer potential with a distribution having integrable first derivatives, and a part which is regular. These properties are sufficient

to guarantee uniform continuity in $\overline{S}$, and an estimate of the form

$$||H\vec{h}(\vec{x}+\vec{y}) - H\vec{h}(\vec{x})||_{\gamma} \leq \delta ||\vec{h}||_{\gamma}$$

for any δ , provided γ is sufficiently small.⁷ This proves the equicontinuity of $\{H\vec{h}_i, i = 1, 2, \dots\}$.

The above results tell us, first of all, that H has a real discrete spectrum whose only possible point of accumulation is the origin $\mu = 0$. By reflection it is easy to show that $\vec{h}(-\vec{x})$ is an eigenfunction of H associated with the eigenvalue $-\mu$ whenever $\vec{h}(\vec{x})$ is an eigenfunction associated with μ . Thus the spectrum of H is also symmetric. (As remarked in Section 1, we are here concerned only with the positive spectrum.) Clearly any eigenfunction of H acting on C^1 is also an eigenfunction of H acting on $\mathcal{H}$. Conversely, if $\vec{h}$ is an eigenfunction of H acting on $\mathcal{H}$, there is a natural extension of $\vec{h}$ to the region $x > 1$, defined by

$$\vec{h}^* = \mu^{-1} H\vec{h}$$

7. See e.g. H. Jeffreys and B.S. Jeffreys, Mathematical Physics (Cambridge University Press, 1956), Chapter 6.

where μ is the associated eigenvalue. It is not difficult to show by the methods used above and the properties of $\mathbf{K}$ that this extension is continuously differentiable at each point in $x < 1$, and hence that $h \in C^1$. In particular any $h \in C^1$ can be represented by an eigenfunction expansion, in the eigenfunctions of H^2 , whose first derivatives converge absolutely and uniformly in $\bar{S}$. A complete set of eigenfunctions of H in $\mathcal{H}$ will be exhibited in Section 6.

We summarize the results of this Section in Lemma 1: The transformations L, M, H, J and the kernels $\mathbf{E}, \mathbf{K}$ have the following properties (i) The decompositions

$$K_{\alpha\beta} = \frac{\varepsilon_{\alpha\beta\gamma}}{4\pi} \frac{x_\gamma - \xi_\gamma}{r^3} + F_{\alpha\beta} \quad (17a)$$

$$E_{\alpha\beta} = -\frac{1}{4\pi} \frac{\partial^2}{\partial x_\alpha \partial \xi_\beta} \frac{1}{r} + G_{\alpha\beta} \quad (17b)$$

hold, where $\mathbf{F}$ and $\mathbf{G}$ are regular functions of $\vec{x}, \vec{\xi}$ in S . (ii) H is self-adjoint on $\mathcal{H}$ and compact on $C^1(\bar{S})$, and H^2 is compact on $\mathcal{H}$. (iii) If $\vec{h}, \emptyset \in C^1$ solves $L\vec{h} = \lambda\vec{h}$, $M\vec{h} = \lambda\emptyset$ then $\vec{h}, \emptyset$ is a solution of (3), (4). If $\vec{h}, \emptyset \in C^1$ solves $H\vec{h} = \mu\vec{h}$, $J\vec{h} = \mu\emptyset$ then $\vec{h}, \emptyset$ is a solution of (3), (4) provided $\mu^{-1} \vec{h}$ replaces $\lambda^{-1} \vec{q} \times \vec{h}$ in (3a).

3. The Perturbation Problem. Main Results.

In the present Section we shall assume that H possesses a positive isolated point eigenvalue μ of odd multiplicity m . We shall make use of the following essential result, which will be proved in Sections 4 and 5: Theorem 1. With L, H defined as above, there is a one parameter family of operators $\hat{H}(c) = L\hat{L}(c)$ such that for any $c \neq 0$ $\|\hat{H} - H\|_\gamma = o(1)$ as $\varepsilon \rightarrow 0$ for every γ satisfying $\frac{1}{2} < \gamma < 1$.

On this basis we now investigate for fixed $c \neq 0$ the perturbation of μ caused by the addition of the "small" operator $V = \hat{H} - H$. If $(H - \nu)^{-1}$ exists we may write

$$\hat{H}\vec{h} = (H + V)\vec{h} = \nu\vec{h}$$

in a form analogous to the operational form utilized in Part I, namely

$$\vec{h} = (H - \nu)^{-1} V\vec{h} = T\vec{h}$$

Suppose now that $\{\vec{f}_i, i = 1, 2, \dots, m\}$ is the set of linearly independent eigenfunctions of H in C^1 associated with μ . Since this sequence is square-integrable over S we may assume that

$$\int_S f_i^2 d\vec{\xi} = 1, \quad i = 1, 2, \dots, m, \quad \int_S \vec{f}_i \cdot \vec{f}_j d\vec{\xi} = 0, \quad i \neq j$$

and define the projection operators on C^1

$$P_i \vec{h} = \left(\int_S \vec{f}_i \cdot \vec{h} d\vec{\xi} \right) \vec{f}_i, \quad \vec{x} \in S$$

$$P = \sum_{i=1}^m P_i$$

As in I, we look for a solution of $\vec{h} = T\vec{h}$ of the form

$$\vec{g} = (I - \tilde{T})^{-1} \vec{f}$$

where

$$\tilde{T} = T - PT$$

and now

$$\vec{f} = \sum_{i=1}^m \alpha_i(\varepsilon) \vec{f}_i$$

where the $\alpha(\varepsilon)$ are undetermined functions of ε presumably defined on some ε_0 -interval, $0 < \varepsilon \leq \varepsilon_0 > 0$. For compatibility we must now satisfy the system of m equations

$$\alpha_i \vec{f}_i = P_i T (I - \tilde{T})^{-1} \vec{f}, \quad i = 1, 2, \dots, m \quad (18)$$

or, multiplying through by $H - \nu$ and taking coefficients

$$(\mu - \nu)\alpha_i = \sum_{j=1}^m A_{ij}\alpha_j \quad (19)$$

where

$$A_{ij} = \int_S \vec{F}_i \cdot [V(I - \tilde{T})^{-1} \vec{F}_j] d\xi$$

Now for any positive ε for which $(I - \tilde{T})^{-1}$ exists, (19) will have a non-trivial solution $\{\alpha_i, i = 1, 2, \dots, m\}$ if and only if

$$\text{Det} \{(\nu - \mu)\delta_{ij} + A_{ij}\} = (\nu - \mu)^m + \psi(\nu, \mu, \varepsilon) = 0 \quad (20)$$

Furthermore, since m is odd, if there is an interval $I(\mu) = [\mu - \delta > 0, \mu + \delta]$ such that for ε small ψ is continuous and arbitrarily small in I , then (20) must have at least one real root in $I(\mu)$, and this root provides a positive eigenvalue of L^2 .

To justify these formal steps, we must show that $(I - \tilde{T})^{-1}$ exists and that the necessary conditions on ψ are satisfied if ε is sufficiently small. Let us first choose δ , with $\mu - \delta > 0$, so that if $\nu \in \Delta = [\mu - \delta, \mu + \delta]$ and $\nu \neq \mu$ then ν is not in the spectrum of H and ν^2 is not in the spectrum of H^2 . Consider then

$$\begin{aligned}\tilde{T}\vec{h} &= (I - P)(H - \nu)^{-1} \nu \vec{h} \\ &= (I - P)(H^2 - \nu^2)^{-1} (H + \nu)\nu \vec{h}\end{aligned}$$

when $\nu \in \Delta$, $\nu \neq \mu$. We first observe that if $\vec{h} \in C^1$ then so is $(H + \nu)\vec{h}$ so that the extension of the latter to $x < 1$ can be represented by an expansion in the eigenfunctions of H^2 , with first derivatives converging absolutely and uniformly in $\bar{S}$. Suppose that $\vec{f}$ is an eigenfunction of H^2 associated with the eigenvalue μ^2 , then $(H - \mu)(H + \mu)\vec{f} = 0$ and either $(H + \mu)\vec{f} = 0$ or else $(H + \mu)\vec{f}$ is an eigenfunction of H associated with μ and therefore $(I - P)(H + \mu)\vec{f} = 0$. Thus if d is the minimum distance between Δ and an eigenvalue of H^2 we have for $\nu \neq \mu$

$$\begin{aligned}|| (I - P)(H^2 - \nu^2)^{-1} (H + \nu)\nu \vec{h} ||_\gamma &< \frac{3}{d} || (H + \nu)V ||_\gamma || \vec{h} ||_\gamma \\ &< \frac{3}{d} [|| H ||_\gamma + \mu + \delta] || V ||_\gamma || \vec{h} ||_\gamma\end{aligned}$$

Since the last estimate is independent of ν we let $\nu \rightarrow \mu$ and obtain an estimate on $\tilde{T}$ which is uniform in Δ . If now γ is chosen in accordance with Theorem 1, for this value $(I - \tilde{T})^{-1}$ exists if ε is sufficiently small.

In a similar way we may show that in the same γ -norm $|\psi|$ can be made arbitrarily small on Δ . Indeed,

this follows immediately from the fact that

$$|A_{ij}| < \frac{4}{3} \pi ||\vec{f}_i||_{\gamma} ||\vec{f}_j||_{\gamma} ||V(I - \tilde{T})^{-1}||_{\gamma}$$

Finally, we must show that in some ε_0 -interval ψ is a continuous function of ν uniformly on Δ . But this follows from the fact that $(I - \tilde{T})^{-1}$ may be differentiated with respect to ν if ε is sufficiently small. Indeed

$$\frac{d}{d\nu} (I - \tilde{T})^{-1} = (I - \tilde{T})^{-2} (I - P) \frac{(H + \nu)^2}{(H^2 - \nu^2)^2} V$$

and the γ -norm of this operator may again be made arbitrarily small. It can then be shown as before that

$$|\frac{d}{d\nu} A_{ij}|$$

is uniformly small on Δ and consequently that ψ is uniformly continuous there.

By using Theorem 1 we have therefore proved
Theorem 2: If H has a positive eigenvalue of odd multiplicity, then for each $c \neq 0$ the operator $\hat{H}(c)$ has a corresponding positive eigenvalue ν . Moreover $(\nu - \mu) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We remark that if the multiplicity

of the unperturbed eigenvalue is even (which may well occur in conductors which are not invariant under arbitrary rotations about a point) the possibility exists that under the perturbation the eigenvalue splits into complex conjugate pairs, so that our method of proof cannot be expected to apply. Indeed, the present methods have aimed only at the existence of the eigenvalue. A more detailed analysis of the solutions of (19) would be needed in order to establish properties of the perturbed eigenfunction. For example, it is not yet known whether or not the perturbed eigenfunction is uniformly continuous in ε as $\varepsilon \rightarrow 0$.

Our main result for L is a simple corollary to Theorem 3: Let $v(c)$, $\vec{h}(c)$ be respectively the real eigenvalue and an associated eigenfunction for $\hat{H}(c)$ as c ranges over Δ . Then there is a point $v_0 \in \Delta$ such that, if $\vec{h}_0 = \vec{h}(v_0)$ $\hat{H}(v_0)\vec{h}_0 = L^2 \vec{h}_0 = v_0 \vec{h}_0$. This property of $\hat{H}$, which is proved in Section 5, implies that $(L - \sqrt{v_0})(L + \sqrt{v_0})\vec{h}(v_0) = 0$, so that one of the two numbers $\pm \sqrt{v_0}$ is the desired real eigenvalue of L .

4. Preliminary Estimates.

In the present Section we derive some preliminary estimates on L and $L^2 - H$ which provide a framework for the proof of Theorem 1. We shall briefly relax our restriction on $\vec{q}$ to motions of the form (1) in order to

exhibit some more general properties of L . In particular, let us alter (1) by allowing the $\vec{v}(\vec{x})$ to be any periodic Beltrami field of even degree ≥ 2 . For this more general class of $\vec{q}$, the orders of magnitude associated with the action of L on C^1 can be studied qualitatively if one assumes the effect of the periodic part to be dominant and argues from the results of I. For spatially-periodic motion in an infinite conductor, the operator T introduced in I acts on a vector in C^1 which is independent of ϵ (i.e. "slowly-varying"), with maximum amplitude 1, to produce a field which varies on a scale of order ϵ (i.e. is "rapidly-varying") and has maximum amplitude $O(\epsilon^{1/J})$. The gradient of the latter is thus $O(\epsilon^{1/J - 1})$. If, however, T operates on a field of order unity which varies on a scale of order ϵ , the result will in general consist of two parts, a "slowly-varying" term of order $\epsilon^{1/J - 1}$ and a "rapidly varying" term of order $\epsilon^{1/J}$. We may express these estimates in a different way by writing for arbitrary $\vec{h} \in C^1$

$$||T\vec{h}|| = O(\epsilon^{1/J} ||\vec{h}||) + O(\epsilon^{1/J} ||\nabla\vec{h}||) \quad (21a)$$

$$||\nabla T\vec{h}|| = O(\epsilon^{1/J - 1} ||\vec{h}||) + O(\epsilon^{1/J} ||\nabla\vec{h}||) \quad (21b)$$

Suppose now that, using (21), we attempt to find a γ -norm which makes T uniformly bounded on C^1 . It is easily seen that we must have

$$O(\varepsilon^{1/J} ||\vec{h}||) + O(\varepsilon^{\gamma + 1/J - 1} ||\vec{h}||) + O(\varepsilon^{1/J} ||\nabla \vec{h}||) \\ + O(\varepsilon^{\gamma + 1/J} ||\nabla \vec{h}||) = O(||\vec{h}||_\gamma)$$

so that necessarily $1 - 1/J \leq \gamma \leq 1/J$. Thus $J = 2$ and $\gamma = 1/2$. We thus deduce that uniform boundedness of T in some γ -norm is a constraint which restricts the motions to fields of degree 2, even though periodic dynamos exist in principle for arbitrary even degree.

Thus result, although somewhat surprising in view of the results of I, is not altogether unexpected. In general the results that can be derived by the methods of functional analysis depend strongly upon the norm used, and in our case it appears that the γ -norms are closely associated with the degree two periodic fields. Presumably a different choice of norm will allow a larger class. For example, it may be necessary to go into one of the spaces C^j , $j > 1$ in order to treat fields with $J > 2$.

Results analogous to (21) can be given for L on $C^1(\bar{S})$: Lemma 1: If $\vec{h} \in C^1$ and if $\vec{v}$ has degree J in (1) then

$$||L\vec{h}|| = O(\varepsilon^{1/J} \log \varepsilon ||\vec{h}||) + O(\varepsilon^\beta ||\vec{h}||) + O(\varepsilon^{1/J} ||\nabla\vec{h}||) \quad (22a)$$

$$||\nabla L\vec{h}|| = O(\varepsilon^{1/J - 1} ||\vec{h}||) + O(\varepsilon^{1/J} \log \varepsilon ||\nabla\vec{h}||) \quad (22b)$$

Apart from the weak $\log \varepsilon$ factor s, when $\beta = 1/2$, (21) and (22) are identical, and it is easily checked that L is uniformly bounded in the γ -norm if $J = 2$, $\gamma = 1/2$, $\beta \geq 0$.

We shall restrict our proof of (22) to the particular case $J = 2$, and return now to the particular motions (1), (2).

We begin by dividing the domain of integration into two parts

$$L\vec{h} = \int_{S'_1} \mathbf{K} \cdot \vec{q} \times \vec{h} \, d\vec{\xi} + \int_{S'} \mathbf{K} \cdot (\vec{q} \times \vec{h}) d\xi = I_1 + I_2 \quad (23)$$

say, where S_α ($\alpha > 0$) is the sphere of radius ε^α and center at some point $\vec{x} \in \bar{S}$, $S'_1 = S_1 \cap S$, $S^1 = S - S'_1$ (cf. Figure 2 with $\alpha = 1$). Next, we split I_1 into two parts according to (15a):

$$I_1 = \frac{1}{4\pi} \int_{S'_1} \nabla_x \frac{1}{r} \times (\vec{q} \times \vec{h}) d\vec{\xi} + \int_{S'_1} \mathbf{F} \cdot (\vec{q} \times \vec{h}) d\vec{\xi} = I_{11} + I_{12}$$

By (1), (2) and the hypothesis on $\vec{w}$ we have $||\vec{q}|| = O(\varepsilon^{-1/2})$, and since $\mathbf{F}$ is regular in $\bar{S}$ we obtain

$$||I_{11}|| = o(\varepsilon^{1/2} ||\vec{h}||) \quad (24)$$

$$||I_{12}||, ||\nabla I_{12}|| = o(\varepsilon^{5/2} ||\vec{h}||) \quad (25)$$

In deriving (25b), and in the remainder of this Section we always keep the boundary of S_α fixed in forming differentials.

To estimate ∇I_{11} we again write

$$I_{11} = \frac{1}{4\pi} \int_{S_1'} \frac{\nabla \times (\vec{q} \times \vec{h})}{r} d\vec{\xi} + \frac{1}{4\pi} \int_{\partial S_1'} \frac{(\vec{q} \times \vec{h}) \times \vec{n}}{r} ds$$

and obtain

$$||\nabla I_{11}|| = o(\varepsilon^{-1/2} ||\vec{h}||) + o(\varepsilon^{1/2} ||\nabla \vec{h}||) \quad (26)$$

Now consider the volume integral I_2 . Using (1) and the vector identity

$$(\nabla \times \vec{A}) \times \vec{B} = -(\nabla \times \vec{B}) \times \vec{A} + \vec{A} \cdot \nabla \vec{B} + \vec{B} \cdot \nabla \vec{A} - \nabla(\vec{A} \cdot \vec{B})$$

we may integrate by parts to obtain

$$\begin{aligned}
I_2 = & \epsilon^\beta \int_{S'} \mathbf{K} \cdot (\vec{\omega} \times \vec{h}) d\vec{\xi} - \epsilon \int_{\partial S'} \omega \mathbf{K} \cdot [\vec{v}(\vec{h} \cdot \vec{n}) + \vec{h}(\vec{v} \cdot \vec{n}) - \vec{n}(\vec{v} \cdot \vec{h})] ds \\
& - \epsilon \int_{S'} \omega \mathbf{K} \cdot [\vec{v} \times (\nabla \times \vec{h}) - \vec{v} \nabla \cdot \vec{h}] \\
& + \epsilon \int_{S'} \omega [(\vec{v} \cdot \nabla \mathbf{K}) \cdot \vec{h} + (\vec{h} \cdot \nabla \mathbf{K}) \cdot \vec{v}] d\vec{\xi} \\
& + \epsilon \int_{S'} \mathbf{K} \cdot \vec{h} (\vec{v} \cdot \nabla \omega) d\vec{\xi} = \sum_{i=1}^5 I_{2i}
\end{aligned}$$

Now since any component of $\mathbf{K}$ and $\nabla \mathbf{K}$ have absolute bounds of the form Cr^{-2} and Cr^{-3} respectively we have clearly

$$||I_{21}|| = O(\epsilon^\beta ||\vec{h}||) \quad (27a)$$

$$||I_{22}|| = O(\epsilon^{1/2} ||\vec{h}||) \quad (27b)$$

$$||I_{23}|| = O(\epsilon^{1/2} ||\nabla \vec{h}||) \quad (27c)$$

$$||I_{24}|| = O(\epsilon^{1/2} \log \epsilon ||\vec{h}||) \quad (27d)$$

and, using (2),

$$||I_{25}|| = O(\epsilon^{1/2} ||\vec{h}||) \quad (27e)$$

Forming the gradients, we also obtain easily

$$||\nabla I_{21}|| = O(\epsilon^\beta \log \epsilon ||\vec{h}||) \quad (28a)$$

$$||\nabla I_{22}|| = O(\epsilon^{-1/2} ||\vec{h}||) \quad (28b)$$

$$||\nabla I_{23}|| = O(\epsilon^{1/2} \log \epsilon ||\vec{h}||) \quad (28c)$$

$$||\nabla I_{24}|| = O(\epsilon^{-1/2} ||\vec{h}||) \quad (28d)$$

$$||\nabla I_{25}|| = O(\epsilon^{1/2} \log \epsilon ||\vec{h}||) \quad (28e)$$

We next turn to an estimate on $L^2 - H$. Lemma 2:
Let $\vec{q}$ be given by (1), (2). Then if $\vec{h} \in C^1$ and if α is
any number satisfying $3/4 < \alpha < 1$ then we have the
estimate

$$\begin{aligned} ||L^2 \vec{h} - H\vec{h}|| &= O(\epsilon^{1-\alpha} ||\vec{h}||) + O(\epsilon^{2\beta} ||\vec{h}||) \\ &+ O(\epsilon^{2\alpha-1} ||\nabla \vec{h}||) \end{aligned} \quad (29)$$

To obtain (29), we begin by dividing the integral as before, now choosing α in the definition of S_α to lie between $3/4$ and 1 . We indicate this change by adding the superscript (α) to the various sub-integrals. The estimates for $||I_{21}^{(\alpha)}||$ carry over, as do (28a), (28c), and (28e). However (28b) and (28d) may be replaced by

$$||\nabla I_{22}^{(\alpha)}||, ||\nabla I_{24}^{(\alpha)}|| = O(\epsilon^{1/2 - \alpha} ||\vec{h}||) \quad (30)$$

For the integrals over S'_α we now have

$$||I_{11}^{(\alpha)}|| = O(\epsilon^{\alpha - 1/2} ||\vec{h}||) \quad (31a)$$

$$||I_{12}^{(\alpha)}||, ||\nabla I_{12}^{(\alpha)}|| = O(\epsilon^{3\alpha - 1/2} ||\vec{h}||) \quad (31b)$$

$$||\nabla I_{11}^{(\alpha)}|| = O(\epsilon^{\alpha - 3/2} ||\vec{h}||) + O(\epsilon^{\alpha - 1/2} ||\nabla \vec{h}||) \quad (32)$$

Applying now (22a) (with $J = 2$) and using (27), (28a), (28c), (28e), (30) we obtain

$$\begin{aligned} ||LI_2^{(\alpha)}|| &= O(\epsilon^{1 - \alpha} ||\vec{h}||) + O(\epsilon^{2\beta} ||\vec{h}||) \\ &+ O(\epsilon \log \epsilon ||\nabla \vec{h}||) \end{aligned} \quad (33)$$

Using (31b), we also have

$$||LI_{12}^{(\alpha)}|| = O(\epsilon^{3\alpha} \log \epsilon ||\vec{h}||) \quad (34)$$

Lastly, we consider $LI_{11}^{(\alpha)}$ and utilize the following vector mean-value theorem: If $\vec{h} \in C^1$ then

$$\vec{h}(\vec{\xi}) = \vec{h}(\vec{x}) + \vec{r} \cdot \nabla \vec{h}(\vec{x}_0)$$

for some $\vec{x}$ in $\bar{S}$ on the line joining $\vec{x}$ and $\vec{\xi}$. Making this substitution in $I_{11}^{(\alpha)}$ we obtain

$$\begin{aligned}
I_{11}^{(\alpha)} &= \frac{1}{4\pi} \int_{S'_\alpha} \nabla_x \frac{1}{r} \times [\vec{q}(\vec{\xi}) \times \vec{h}(\vec{x})] d\vec{\xi} \\
&+ \frac{1}{4\pi} \int_{S'_\alpha} \nabla_x \frac{1}{r} \times [\vec{q}(\vec{\xi}) \times \vec{r} \cdot \nabla \vec{h}(\vec{x}_0)] d\vec{\xi} \\
&= I_{111}^{(\alpha)} + I_{112}^{(\alpha)}
\end{aligned}$$

Now $||I_{112}^{(\alpha)}|| = O(\varepsilon^{2\alpha - 1/2} ||\nabla \vec{h}||)$ and therefore

$$||LI_{112}^{(\alpha)}|| = O(\varepsilon^{2\alpha - 1} ||\nabla \vec{h}||) \quad (35)$$

Turning now to $I_{111}^{(\alpha)} = Q(\vec{q})$ we note that

$$||Q(\vec{w})|| = O(\varepsilon^\alpha ||\vec{h}||)$$

So that

$$\varepsilon^\beta ||LQ(\vec{w})|| = O(\varepsilon^{\alpha + \beta - 1/2} ||\vec{h}||) \quad (36)$$

where the right-hand side is $O(\varepsilon^{2\beta} ||\vec{h}||)$ if $0 < \beta \leq 1/4$ and is $O(\varepsilon^{1-\alpha} ||\vec{h}||)$ if $\beta > 1/4$. Also from (2) we deduce that

$$||Q(\varepsilon \nabla \omega \times \vec{v})|| = O(\varepsilon^{\alpha - 1/2} ||\vec{h}||)$$

Furthermore $Q(\epsilon \nabla \omega \times \vec{v}) = 0$ when the distance from $\vec{x}$ to B exceeds $\epsilon + \epsilon^\alpha$. Hence

$$||LQ(\epsilon \nabla \omega \times \vec{v})|| = O(\epsilon^{2\alpha - 1} ||\vec{h}||) \quad (37)$$

The contribution to L^2 of order unity is obtained from $LQ(\omega \vec{v})$. Now

$$Q(\omega \vec{v}) = Q(\epsilon^{1/2} \vec{v}) - Q((\epsilon^{-1/2} - \omega) \vec{v})$$

and

$$||LQ(\epsilon^{-1/2} - \omega) \vec{v})|| = \epsilon^\alpha ||h|| = O(\epsilon^{1 - \alpha} ||\vec{h}||) \quad (38)$$

if $\alpha > 3/4$. Also S'_α will be a sphere of radius ϵ^α except when $\vec{x}$ lies within a distance ϵ^α from B.

Thus the error that is made in the computation of $LQ(v)$ by assuming that S'_α is a sphere is at most $O(\epsilon^{2\alpha - 1} ||\vec{h}||)$ and can be absorbed into the right-hand side of (37). Consider then

$$\begin{aligned} I_c &= \epsilon^{-1} v(x) \times \int_{\rho < \epsilon^\alpha} \frac{\vec{\rho}}{\rho^3} \times [\vec{v}(\vec{x} + \vec{\rho}) \times \vec{h}(\vec{x})] d\vec{\rho} \\ &= \epsilon^{-1} \int_{\rho < \epsilon^\alpha} \left\{ [\vec{\rho} \cdot \vec{h}(\vec{x})] [\vec{v}(\vec{x}) \times \vec{v}(\vec{x} + \vec{\rho})] - [\vec{\rho} \cdot \vec{v}(\vec{x} + \vec{\rho})] [\vec{v}(\vec{x}) \times \vec{h}(\vec{x})] \right\} \frac{d\vec{\rho}}{\rho^3} \end{aligned}$$

It is not difficult to see, from the fact that $\vec{v}$ is solenoidal, that the second term in the braces gives no contribution to the integral. Carrying out the integration of the first term, we obtain

$$I_c = \vec{h} - \eta(\varepsilon)\vec{h} + R \quad (39)$$

where

$$\eta(\varepsilon) = \varepsilon^{1-\alpha} \sin \varepsilon^{\alpha-1} \quad (40)$$

and R is a linear combination of terms of the form

$$\vec{i}_\alpha h_\beta(\vec{x}) \left\{ \begin{array}{c} \sin \left(\frac{\vec{k} \cdot \vec{x}}{\varepsilon} \right) \\ \cos \left(\frac{\vec{k} \cdot \vec{x}}{\varepsilon} \right) \end{array} \right\}, \quad \vec{k} \neq 0$$

Now it is a simple matter to show, by retracing the proof of (22a) that

$$||LR|| = O(\varepsilon^{1/2 + \beta} ||\vec{h}||) + O(\varepsilon ||\nabla \vec{h}||) \quad (41)$$

Combining (33) - (41) we obtain (29).

5. The Modified Operator.

It follows that for any γ satisfying $1/2 < \gamma < 1$ there is a corresponding α such that (29) implies

$$||L^2\vec{h} - H\vec{h}|| = o(||\vec{h}||_\gamma)$$

We now ask whether or not the estimates given in Section 4 allow the stronger statement

$$||L^2\vec{h} - H\vec{h}||_\gamma = o(||\vec{h}||_\gamma) \quad (42)$$

for some γ satisfying $1/2 < \gamma < 1$. That this is not the case can be shown by considering $||\nabla L I_{23}^{(\alpha)}||$. We recall that

$$I_{23}^{(\alpha)} = -\varepsilon \int_{S'} \omega \kappa [\vec{v} \times (\nabla \times \vec{h}) - \vec{v} \nabla \cdot \vec{h}] d\xi \quad (43)$$

Using (27c) and (28c) with $J = 2$ in (22b) we obtain only

$$||\nabla L I_{23}^{(\alpha)}|| = o(||\nabla h||)$$

which does not allow (42).

To remedy this situation we introduce an intermediate modified operator $\tilde{L}$ obtained as follows. Dividing the expression for $L\vec{h}$ into subintegrals about S'_α as in Section 4, we alter $I_{23}^{(\alpha)}$ by assuming that $\vec{h}$ satisfies $L\vec{h} = \vec{h}$. In that case

$$\nabla \cdot \vec{h} = 0 \quad \nabla \times \vec{h} = \nabla M \vec{h} + \vec{q} \times \vec{h}$$

and $I_{22}^{(\alpha)}$ is changed to

$$\tilde{I}_{23}^{(\alpha)}(\vec{h}) = -\varepsilon \int_{S'} \omega \mathbf{K} \cdot [\vec{v} \times \nabla M \vec{h} + \vec{v} \times (\vec{q} \times \vec{h})] d\xi \quad (44)$$

We now define the modified operator $\tilde{L}$ by

$$\tilde{L}\vec{h} = I_1^{(\alpha)}(\vec{h}) + \sum_{i \neq 3} I_{2i}^{(\alpha)}(\vec{h}) + \hat{I}_{23}^{(\alpha)}(\vec{h}) \quad (45)$$

It should be noted that (45) defines a family of operators $\tilde{L}(\alpha)$. However the dependence on α will eventually be suppressed since it will be fixed along with γ .

Consider now $||\nabla L\tilde{L}||$. From (44) we have

$$||\tilde{I}_{23}^{(\alpha)}|| = o(\varepsilon^{1/2} ||\nabla M \vec{h}||) + o(||\vec{h}||) \quad (46)$$

Using now the fact that $\vec{\phi}$ and $\mathbf{K}$ have singularities of the same order at $\vec{x} = \vec{\xi}$ it follows from the proof of (22b) that

$$||\nabla \vec{Mh}|| = O(\varepsilon^{-1/2} ||\vec{h}||) + O(\varepsilon^{1/2} \log \varepsilon ||\nabla \vec{h}||) \quad (47)$$

Combining (46) and (47) we have

$$||\tilde{I}_{23}^{(\alpha)}|| = O(||\vec{h}||) + O(\varepsilon \log \varepsilon ||\nabla \vec{h}||) \quad (48)$$

For $||\nabla \tilde{I}_{23}^{(\alpha)}||$ we now obtain

$$||\nabla \tilde{I}_{23}^{(\alpha)}|| = O(\log \varepsilon ||\vec{h}||) + O(\varepsilon (\log \varepsilon)^2 ||\nabla \vec{h}||) \quad (49)$$

Using (48) and (49) in (22), it then follows that

$$\begin{aligned} ||L \tilde{I}_{23}^{(\alpha)}|| &= O(\varepsilon^{1/2} \log \varepsilon ||\vec{h}||) + O(\varepsilon^\beta \log \varepsilon ||\vec{h}||) \\ &\quad + O(\varepsilon^{3/2} (\log \varepsilon)^2 ||\nabla \vec{h}||) + O(\varepsilon^{\beta+1} \log \varepsilon ||\nabla \vec{h}||) \end{aligned} \quad (50)$$

$$||\nabla L \tilde{I}_{23}^{(\alpha)}|| = O(\varepsilon^{-1/2} ||\vec{h}||) + O(\varepsilon^{1/2} \log \varepsilon ||\nabla \vec{h}||) \quad (51)$$

Moreover, from (27), (28), (31b) and (22b) we obtain

$$||\nabla L(I_2^{(\alpha)} - I_{23}^{(\alpha)})|| = O(\log \varepsilon ||\vec{h}||) \quad (52)$$

$$||\nabla L I_{12}^{(\alpha)}|| = O(\varepsilon^{3\alpha-1} ||\vec{h}||) \quad (53)$$

Our final estimate is on $|| \text{LI}_{11}^{(\alpha)} ||$. Here it is sufficient to use (31a) and (32) in (22b), giving

$$|| \nabla \text{LI}_{11}^{(\alpha)} || = O(\varepsilon^{\alpha-1} \log \varepsilon || \vec{h} ||) + O(\varepsilon^{\alpha} \log \varepsilon || \nabla \vec{h} ||) \quad (54)$$

Thus, from (45) - (49) we deduce that

$$|| \vec{Lh} ||_{\gamma} = O(|| \vec{h} ||_{\gamma}) , \quad 1/2 < \gamma < 1 \quad (55)$$

and, combining (29) with (50) - (54), that

$$\begin{aligned} || \text{L}\vec{Lh} - \vec{h} || &= [O(\varepsilon^{1-\alpha}) + O(\varepsilon^{\beta} \log \varepsilon) \\ &+ O(\varepsilon^{\gamma-1/2}) + O(\varepsilon^{2\alpha-1+\gamma})] || \vec{h} ||_{\gamma} , \quad 1/2 < \gamma < 1 \end{aligned} \quad (56)$$

From (56) it follows that if $\beta > 0$ and γ, α satisfy $1/2 < \gamma < 2\alpha - 1$, $3/4 < \alpha < 1$, that the right-hand side is $O(\varepsilon^s || \vec{h} ||_{\gamma})$ for some $s > 0$. In particular this latter estimate holds for any γ satisfying $1/2 < \gamma < 1$.

We now define the family of modified operators $\hat{L}(c)$ for $c \neq 0$ by the implicit equation

$$\hat{L}(c)\vec{h} = I_1^{(\alpha)}(\vec{h}) + \sum_{l \neq 3} I_{2l}^{(\alpha)}(\vec{h}) + \frac{1}{c} \tilde{I}_{23}^{(\alpha)}(\hat{L}\vec{h}) \quad (57)$$

From (50), (51), and (55) we have

$$||\widehat{Lh}||_{\gamma} \leq o(||\vec{h}||_{\gamma}) + o(\varepsilon^{\gamma-1/2} ||\widehat{Lh}||_{\gamma}), \quad 1/2 < \gamma < 1 \quad (58)$$

and (58) shows that $\widehat{L}$ exists (e.g. it can be defined as the limit of a sequence of iterated solutions of (57)) and is a uniformly bounded operator on C^1 provided $1/2 < \gamma < 1$. Defining

$$\widehat{H}(c) = L\widehat{L}(c)$$

and using (57) and (58) we obtain

$$||\widehat{H}-H||_{\gamma} = o(||L\vec{L}-H||_{\gamma}), \quad c \neq 0, \quad 1/2 < \gamma < 1 \quad (59)$$

which, together with (56), implies Theorem 1.

We now turn to the proof of Theorem 3, and restrict c to lie in the interval $\Delta = [\mu - \delta, \mu + \delta]$. On this interval the perturbation operator V is clearly continuous in c so that for all sufficiently small values of ε the real eigenvalue $v(c)$ of $\widehat{H}$ obtained by Theorem 2 satisfies

$$v(c) = \mu + \rho(\varepsilon, c)$$

where ρ is continuous in c and uniformly small.

It follows that the equation

$$v = \mu + \rho(\varepsilon, v)$$

has a solution $v_0(\varepsilon)$ in Δ for all sufficiently small ε . Let $H(v_0) = H_0(\varepsilon)$, and suppose that $\vec{h}_0(\varepsilon)$ is a corresponding eigenfunction, so that

$$\hat{H}_0 \vec{h}_0 = v_0 \vec{h}_0 \quad 0 < \varepsilon \leq \varepsilon_0 > 0 \quad (60)$$

We then claim that

$$L^2 \vec{h}_0 = v_0 \vec{h}_0$$

Indeed, by the definition of $\hat{L}$ and L it follows that

$$\begin{aligned} \hat{L}(v_0) \vec{h}_0 &= \hat{L}_0(\vec{h}_0) = L \vec{h}_0 + \frac{1}{v_0} \tilde{I}_{23}^{(\alpha)}(\hat{L}_0 \vec{h}_0) - I_{23}^{(\alpha)}(\vec{h}_0) \\ &= L \vec{h}_0 + \frac{1}{v_0} I_{23}^{(\alpha)}(\hat{H}_0 \vec{h}_0) - I_{23}^{(\alpha)}(\vec{h}_0) \\ &= L \vec{h}_0 + I_{23}^{(\alpha)}(\vec{h}_0) - I_{23}^{(\alpha)}(\vec{h}_0) \\ &= L \vec{h}_0 \end{aligned}$$

using (60). Thus $\hat{H}_0 \vec{h}_0 = L \hat{L} \vec{h}_0 = L^2 \vec{h}_0 = v_0 \vec{h}_0$ and the proof of Theorem 3 is complete.

6. The Spectrum of H.

The eigenvalue problem $H\vec{h} = \mu\vec{h}$ can be solved for the geometry shown in Figure 2 by standard means and we omit the details of the computations. A complete (in $\mathcal{H}$) set of eigenfunctions have the following form ($\sigma = \mu^{-1}$):

$$\vec{h} = \nabla\psi_1, \quad \phi = \phi_1, \quad x > 1$$

$$\vec{h} = \nabla\psi_2 + \nabla \times (\vec{x}\psi_2), \quad \phi = \phi_2, \quad a < x < 1$$

$$\vec{h} = \nabla\psi_3 + \nabla\left(\frac{\partial}{\partial x} \times \psi_4\right) + \sigma \nabla \times (\vec{x}\psi_4) + \sigma^2 \vec{x}\psi_4,$$

$$\phi = \phi_3, \quad 0 \leq x < a.$$

where, in terms of the surface harmonics $Y_{mn}(\theta, \phi)$,

$$\psi_1 = x^{-(n+1)} Y_{mn}, \quad \phi_1 = \sigma \frac{(n+1)}{n} \psi_1$$

$$\psi_2 = A(x^n - x^{-(n+1)}) Y_{mn}, \quad \phi_2 = \frac{\partial}{\partial x} (x\psi_2)$$

$$\psi_3 = B x^n Y_{mn}, \quad \phi_3 = -\sigma\psi_3$$

$$\psi_4 = C f_n(x) Y_{mn}$$

$$f_n(x) = x^n \left(\frac{1}{x} \frac{d}{dx}\right)^n \frac{\sin \sigma x}{x} = (-1)^n (\sigma)^{n+1/2} \sqrt{\frac{\pi}{2x}} J_{n+1/2}(\sigma x)$$

and

$$A = \sigma \frac{n+1}{2n^2+n}, \quad B = - \frac{n+1}{2n^2+n} [na^{-(2n+1)}]_{+n+1}]$$

$$C = \frac{n+1}{2n^2+n} \cdot \frac{a^n - a^{-(n+1)}}{f_n(a)}$$

The characteristic equation determining the eigenvalue is found to be

$$(a^{2n+1}-1)[(n+1) f_n(a) + af'_n(a)] = (2n+1) f_n(a)[a^{2n+1}]_{+n/n+1}] \quad (61)$$

Each finite real root of (61) provides a non-zero real eigenvalue μ_n of H having multiplicity $2n+1$. (There are $2n+1$ distinct surface harmonics of order n .) Associated with each order n there is, moreover, a sequence of eigenvalues $\{\mu_n^{(j)}, j = 1, 2, \dots\}$ such that $|\mu_n^{(j)}| \rightarrow 0$ as $j \rightarrow \infty$ and $|\mu_n^{(1)}| \rightarrow 0$ as $n \rightarrow \infty$. If we restrict attention to the positive eigenvalues, we see that the unperturbed eigenfunctions of the present theory depend upon three integer parameters, two which determine the surface harmonic of the vacuum field, and a third which determines the core field. There should also be added a discrete parameter taking values ± 1 which specifies the direction of the lines of force, if it is assumed

that the solutions are renormalized to have core magnetic energy unity. Finally, associated with the dynamos $-\vec{q}(-\vec{x})$ (cf. Section 1) we have another similar family of eigenfunctions of H associated with the negative spectrum.

The above results are easily proved from (61) and well known properties of the spherical Bessel functions $J_{n+1/2}(x)$, using in particular the lower bound $[(n+1/2)(n+5/2)]^{1/2}$ for the smallest positive zero. If we let $a \rightarrow 1$ (a limit process which is non-uniform in n for large n), we see from (61) that the eigenvalues tend to the zeros of $J_{n+1/2}(\sigma a)$. An interesting special case are the eigenvalues for the vacuum dipole solutions, obtained by setting $n = 1$ in (61). We find that σa satisfies in this case

$$\tan(\sigma a) = \frac{\sigma a}{1+b(a)(\sigma a)^2} \quad , \quad b(a) = \frac{2}{3} \frac{1-a^3}{1+2a^3}$$

Thus the sequence $\mu_1^{(j)}$ satisfies

$$\frac{a}{(j+1/2)\pi} < \mu_1^{(j)} < \frac{a}{j\pi} \quad , \quad j = 1, 2, \dots$$

The maximum positive eigenvalue of H is thus bounded above by a/π .

Combining these results with Theorems 2 and 3 we have proved Theorem 4: In a spherical conductor with a concentric spherical core, for each positive eigenvalue μ of H there exists a corresponding real eigenvalue λ of L provided that ϵ is sufficiently small, with $|\lambda^2 - \mu| \rightarrow 0$ as $\epsilon \rightarrow 0$. The same can be said for the negative eigenvalues of H provided $-\vec{q}(-\vec{x})$ replaces $\vec{q}(\vec{x})$ in L .

7. Remarks.

In this report we have discussed the magneto-hydrodynamic dynamo problem for a specific class of incompressible fluid motions in a spherical core, defined by (1), (2). The results provide a strong affirmative answer to the basic existence problem of the kinematic theory. Our operational methods, while basically elementary, are somewhat different from other analytic formulations of the dynamo problem because of the use of a maximum norm on C^1 rather than the inner product norm on $\mathcal{H}$. Such an approach seems to be required when the underlying perturbational theory is based upon a scale separation of the dominant velocity and electromagnetic fields. One significant aspect of the spectral analysis is the selection of an iterate of the exact induction operator L , rather

than the operator itself, for study. Thus, while L^2 is "close" in a sense to H , no analogous statement can be made for L , and indeed it seems unlikely that the motions (1) can lead to any simple comparison operator for L . Physically, the reason for this distinction can be traced to the notion of "smoothing" or "averaging". The nearness of L^2 and H on C^1 is due to the fact that the second iteration averages out large, rapid oscillations of the integrand which occur when ε is small. If L is written as the product HS , where S is the matrix operator $(\overline{q}x)$, then although $L^2 = H(SHS)$, we cannot deduce from our results that SHS is close to the identity operator on C^1 . It is unlikely that this simpler comparison, which omits the final averaging induced by the second operation with H , can replace the estimates actually utilized.

There are some immediate extensions of these results. It can be shown that Theorems 1 and 2 hold under the following weaker conditions: (i) The conductor is a closed starlike region having a smooth boundary. (ii) The core (which may have several components) is orientable, and its boundary is smooth and lies in the interior of the conductor. (iii) The core is such that motions of the form (1), (2) exist,

i.e. a cut-off function ω can be constructed analogous to (2). Theorem 3 then follows provided the operator H has a real eigenvalue of odd multiplicity.

It is also possible to enlarge the class of admissible motions $\vec{q}$. It is sufficient that the periodic part $\vec{v}(\vec{x})$ be an asymptotically regenerative trigonometric Beltrami field of degree 2, with

$$\nabla \times \vec{v} = - \operatorname{sgn}(\mu) \vec{v}(x)$$

$$A_0 = \operatorname{sgn}(\mu) I$$

(cf. I). This can be shown by reconsidering the estimate on $LQ(\omega \vec{v})$ in Section 4 for the general class of periodic fields admitted in I. In a similar way, it is possible to admit motions whose periodic part contains a "slow" dependence on $\vec{x}$ (varying on a length scale $O(1)$). For such fields similar theorems apply, but the operator H must be replaced by an operator of the form $Hm(\vec{x})$ where m , which describes the variation on the large scale of the limit matrix $A_0 = m(\vec{x})I$, is assumed to be independent of ε and continuously differentiable in $\bar{S}$.

It is conjectured that the restriction that S be properly contained in the conductor, and the condition that $\vec{w} = 0$ on B , can be removed. If the core fills the conductor then eigenfunction expansions of the first derivatives of the magnetic field do not converge uniformly on B . Also in this case estimates on I_{24} near B derived in Section 4 must be revised. However, it seems likely that these are technical matters which can be corrected, with the result that existence for arbitrary cores inside the conductor and arbitrary $\vec{w}$ vanishing on B implies existence when core and conductor coincide and $\vec{w}$ satisfies no more than a tangency condition on B .

As a model for the geomagnetic dynamo, the most realistic model within the present theory would be obtained by taking a to be very close to 1. In this case a thin mantle whose conductivity is the same as the core fluid might model the screening effects of a thick, low-conductivity mantle. The effects of variable (and even non-scalar) conductivity in the region $a < x < 1$ could actually be incorporated into the present

analysis without much difficulty; since this amounts essentially only to a modification of the regular parts F, G of the tensors K, E . Finally, we note that the presence of a solid concentric spherical body could be modeled by restricting the fluid motion to the shell $0 < b < x < a < 1$.

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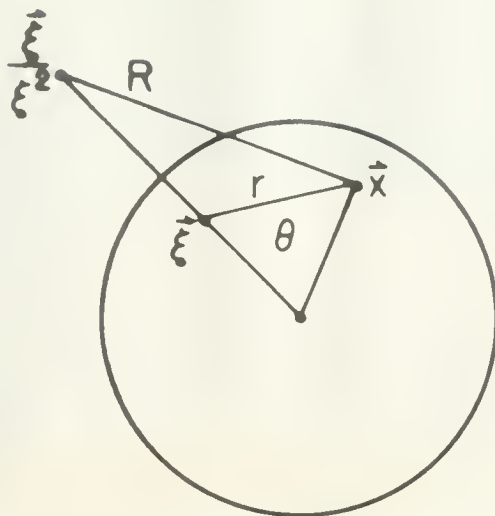


Figure 1
Inversion in the Unit Sphere

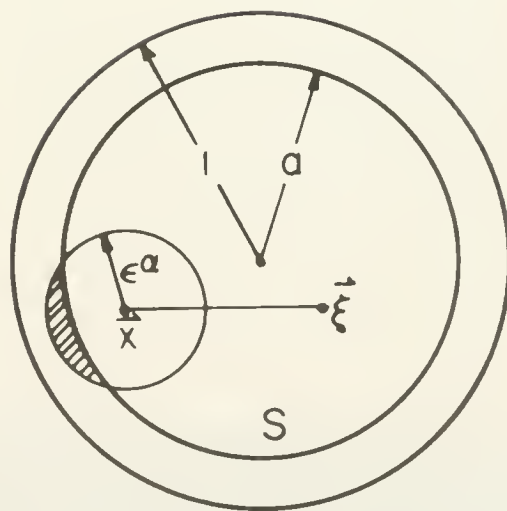


Figure 2
Splitting of the Domain of Integration

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